On the existence of bounded solutions for nonlinear second order neutral nonlinear difference equations

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This paper is dedicated to Gerry Ladas on the occasion of his retirement.

Abstract

We consider the neutral difference equation of the following form

$$\Delta\left(r_n\left(\Delta\left(x_n+p_n x_{n-k}\right)\right)^{\gamma}\right)+q_n x_n^{\alpha}+a_n f(x_n)=0.$$

where $x:\mathbb{N}_0\to\mathbb{R}$, $a,p,q:\mathbb{N}_0\to\mathbb{R}$, $r:\mathbb{N}_0\to\mathbb{R}\setminus\{0\}$, $f\colon\mathbb{R}\to\mathbb{R}$ is a continuous function, and k is a given positive integer, $\gamma\leq 1$ is ratio of odd positive integers, α is a nonnegative constant. Sufficient conditions for the existence of a bounded solution are obtained. Also, stability and asymptotic stability are studied. Some earlier results are generalized.

Keywords Difference equation, measures of noncompactness, Darbo's fixed point theorem, boundedness, stability

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1 Introduction

As it is well known difference equations serve as mathematical models in diverse areas, such as economy, biology, physics, mechanics, computer science, finance, see for example [1], [7]. One of such models is the Emden-Fowler equation which originated in the gaseous dynamics in astrophysics and further was used in the study of fluid mechanics, relativistic mechanics, nuclear physics and in the study of chemically reacting systems, see [18]. In the present paper we study the existence of a bounded solution and its asymptotic behavior to a nonlinear second order difference equation, which can be viewed as a generalization of a discrete Emeden-Fowler equation.

The problem we consider is as follows

$$\Delta\left(r_n\left(\Delta\left(x_n+p_nx_{n-k}\right)\right)^\gamma\right)+q_nx_n^\alpha+a_nf(x_{n+1})=0. \tag{1}$$

where $\gamma\leq 1$ is ratio of odd positive integers, $\alpha\geq 0$, $x:\mathbb{N}_0\to\mathbb{R}$, $a:\mathbb{N}_0\to\mathbb{R}$, $p,r:\mathbb{N}_0\to\mathbb{R}\setminus\{0\}$ and $f:\mathbb{R}\to\mathbb{R}$ is a locally Lipschitz function with no further growth assumptions. Here $\mathbb{N}_0:=\{0,1,2,\dots\}$, $\mathbb{N}_k:=\{k,k+1,\dots\}$ where k is a given positive integer, and $\mathbb{R}$ is a set of all real numbers. By a solution of equation (1) we mean a sequence $x:\mathbb{N}_k\to\mathbb{R}$ which satisfies (1) for every $n\in\mathbb{N}_k$.

For the reader's convenience, we note that the background for difference equations theory can be found in numerous well-known monographs: Agarwal [1], Agarwal, Bohner, Grace and O'Regan [2], Agarwal and Wong [3] Elaydi [7], Kocić and Ladas [8] and Peterson [11].

There has been an interest of many authors to study properties of solutions of second order difference equations; see, for example Bajo and Liz [4], [9], Ladas, Qian and Yan [10], Migda and Migda [12], Migda, Schmeidel and Zbąszyniak [13], Schmeidel [14], and Thandapani, Kavitha and Pinelas [16]–[17].

In this paper, we will use axiomatically defined measures of noncompactness as presented in paper [6] by Banaś and Rzepka.

2 Preliminaries

Let $(E,\|\cdot\|)$ be an infinite-dimensional Banach space. If X is a subset of E, then $\bar{X}$, $ConvX$ denoting the closure and the convex closure of X, respectively. Moreover, we denote by $\mathcal{M}_E$ the family of all nonempty and bounded subsets of E and by $\mathcal{N}_E$ the subfamily consisting of all relatively compact sets.

Definition 1. *A mapping $\mu\colon\mathcal{M}_E\to[0,\infty)$ is called a measure of noncompactness in E if it satisfies the following conditions:*

1^0 $\ker\mu=\{X\in\mathcal{M}_E\colon\mu(X)=0\}\neq\emptyset$ *and* $\ker\mu\subset\mathcal{N}_E$,

2^0 $X\subset Y\Rightarrow\mu(X)\leq\mu(Y)$,

3^0 $\mu(\bar{X})=\mu(X)=\mu(Conv\ X)$,

4^0 $\mu(\alpha X+(1-\alpha)Y)\leq\alpha\mu(X)+(1-\alpha)\mu(Y)$ *for* $0\leq\alpha\leq 1$,

5^0 *If* $X_n \in \mathcal{M}_E,\ \ X_{n+1} \subset X_n,\ \ X_n = \bar{X_n}$ *for* $n = 1, 2, 3, \ldots$
and $\lim\limits_{n\to\infty} \mu(X_n) = 0$ *then* $\bigcap\limits_{n=1}^{\infty} X_n \neq \emptyset$.

The following Darbo's fixed point theorem given in [6] is used in the proof of the main result.

Theorem 1. *Let M be a nonempty, bounded, convex and closed subset of the space E and let $T : M \to M$ be a continuous operator such that $\mu(T(X)) \le k\mu(X)$ for all nonempty subset X of M, where $k \in [0, 1)$ is a constant. Then T has a fixed point in the subset M.*

We consider the Banach space l^∞ of all real bounded sequences $x \colon \mathbb{N}_0 \to \mathbb{R}$ equipped with the standard supremum norm, i.e.

$$\|x\| = \sup_{n\in\mathbb{N}_0} |x_n| \text{ for } x \in l^\infty.$$

Let X be a nonempty, bounded subset of l^∞, $X_n = \{x_n : x \in X\}$ (it means X_n is a set of n-th terms of any sequence belonging to X), and

$$diam\ X_n = \sup\left\{|x_n - y_n| : x, y \in X\right\}.$$

We use a following measure of noncompactness in the space l^∞ (see [5])

$$\mu(X) = \limsup_{n\to\infty} diam\ X_n.$$

3 Main Result

In this section, sufficient conditions for the existence of a bounded solution of equation (1) are derived.

Theorem 2. *Let the ratio of odd positive integers $\gamma \in (0, 1]$, a number $\alpha \ge 0$ be fixed and let k be a fixed positive integer. Assume that*

$$f : \mathbb{R} \to \mathbb{R} \text{ is a locally Lipschitz function,} \tag{2}$$

and that the sequences $r : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$, $a, q : \mathbb{N}_0 \to \mathbb{R}$ satisfy

$$\sum_{n=0}^{\infty} \left(\left|\frac{1}{r_n}\right| \sum_{i=n}^{\infty} |a_i| \right)^{\frac{1}{\gamma}} < +\infty, \ \sum_{n=0}^{\infty} \left(\left|\frac{1}{r_n}\right| \sum_{i=n}^{\infty} |q_i| \right)^{\frac{1}{\gamma}} < +\infty. \tag{3}$$

Let the sequence $p : \mathbb{N}_0 \to \mathbb{R}$ satisfy the following condition

$$-1 < \liminf_{n\to\infty} p_n \leq \limsup_{n\to\infty} p_n < 1. \tag{4}$$

Assume additionally that

$$\sum_{j=1}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i| < +\infty, \; \sum_{i=1}^{\infty} |a_i| < +\infty, \; \sum_{i=1}^{\infty} |q_i| < +\infty. \tag{5}$$

Then, there exists a bounded solution $x : \mathbb{N}_k \to \mathbb{R}$ of equation (1).

Proof. Condition (4) implies that there exist $n_1 \in \mathbb{N}_0$ and a constant $P \in [0, 1)$ such that

$$|p_n| \leq P < 1, \text{for } n \geq n_1. \tag{6}$$

Recalling that remainder of a series is the difference between the n-th partial sum and the sum of a series, we denote by α_n and by β_n remainders of both series in (3). This means that

$$\alpha_n = \sum_{j=n}^{\infty} \left(\left|\frac{1}{r_j}\right| \sum_{i=j}^{\infty} |a_i| \right)^{\frac{1}{\gamma}} \text{ and } \beta_n = \sum_{j=n}^{\infty} \left(\left|\frac{1}{r_j}\right| \sum_{i=j}^{\infty} |q_i| \right)^{\frac{1}{\gamma}}$$

We see by (3) that

$$\lim_{n\to\infty} \alpha_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \beta_n = 0. \tag{7}$$

Fix any number $d > 0$. There exists a constant $M_d > 0$ such that $|f(x)| \leq M_d$ for all $x \in [-d, d]$. Chose a constant C such that

$$0 < C \leq \frac{d - Pd}{\left(2^{\frac{1}{\gamma}-1} (M_d)^{\frac{1}{\gamma}} + 2^{\frac{1}{\gamma}-1} (d^{\alpha})^{\frac{1}{\gamma}} \right)}.$$

By condition (3) there exists a positive integer n_2 such that

$$\alpha_n \leq C \quad \text{and} \quad \beta_n \leq C \tag{8}$$

for $n \geq n_2$, $n \in \mathbb{N}_{n_2} : = \{n_2, n_2 + 1, n_2 + 2, \dots\}$.

Define set B as follows

$$B : = \{(x_n)_{n=0}^{\infty} : |x_n| \leq d, \text{ for } n \in \mathbb{N}_{n_3}\}, \tag{9}$$

where $\mathbb{N}_{n_3} \colon = \{n_3, n_3+1, n_3+2, \dots\}$ and $n_3 = \max\{n_1, n_2\}$. Observe that B is a nonempty, bounded, convex and closed subset of l^∞.

Define a mapping $T\colon B \to l^\infty$ as follows

$$(Tx)_n = -p_n x_{n-k} - \sum_{j=n}^{\infty} \left[\frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f(x_{i+1}) + q_i x_i^\alpha \right) \right]^{\frac{1}{\gamma}}. \tag{10}$$

for any $n \in \mathbb{N}_{n_3}$.

We will prove that the mapping T has a fixed point in B. This proof will follow in several subsequent steps.

Step 1. Firstly, we show that $T(B) \subset B$.
We will use classical inequality

$$(a+b)^s \leq 2^{s-1}\left(a^s + b^s\right),\ a, b > 0,\ s \geq 1$$

and the fact $t \to t^{1/\gamma}$ is nondecreasing. If $x \in B$, then by (10), (6), (9), (8) we have

$$\begin{aligned}
|(Tx)_n| \ &\leq |p_n|\,|x_{n-k}| + \left| \sum_{j=n}^{\infty} \left[\frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f(x_{i+1}) + q_i x_i^\alpha \right) \right]^{\frac{1}{\gamma}} \right| \\
&\leq |p_n|\,|x_{n-k}| + \sum_{j=n}^{\infty} \left[\left| \frac{1}{r_j} \sum_{i=j}^{\infty} \left(a_i f(x_{i+1}) + q_i x_i^\alpha \right) \right| \right]^{\frac{1}{\gamma}} \\
&\leq |p_n|\,|x_{n-k}| + \sum_{j=n}^{\infty} \left[\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} \left(|a_i|\,|f(x_{i+1})| + |q_i|\,|x_i|^\alpha \right) \right]^{\frac{1}{\gamma}}
\end{aligned}$$

$$\leq |p_n| |x_{n-k}| + \sum_{j=n}^{\infty} \left[\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} \left(|a_i| M_d + |q_i| d^{\alpha} \right) \right]^{\frac{1}{\gamma}}$$

$$\leq |p_n| |x_{n-k}| + \sum_{j=n}^{\infty} \left[\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |a_i| M_d + \sum_{i=j}^{\infty} |q_i| d^{\alpha} \right]^{\frac{1}{\gamma}}$$

$$\leq |p_n| |x_{n-k}| + 2^{\frac{1}{\gamma}-1} \sum_{j=n}^{\infty} \left[\left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |a_i| M_d \right)^{\frac{1}{\gamma}} + \left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |q_i| d^{\alpha} \right)^{\frac{1}{\gamma}} \right]$$

$$\leq |p_n| |x_{n-k}| + 2^{\frac{1}{\gamma}-1} (M_d)^{\frac{1}{\gamma}} \sum_{j=n}^{\infty} \left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |a_i| \right)^{\frac{1}{\gamma}}$$

$$+ 2^{\frac{1}{\gamma}-1} (d^{\alpha})^{\frac{1}{\gamma}} \sum_{j=n}^{\infty} \left(\left| \frac{1}{r_j} \right| \sum_{i=j}^{\infty} |q_i| \right)^{\frac{1}{\gamma}}$$

$$\leq Pd + 2^{\frac{1}{\gamma}-1} (M_d)^{\frac{1}{\gamma}} C + 2^{\frac{1}{\gamma}-1} (d^{\alpha})^{\frac{1}{\gamma}} C$$

$$\leq Pd + \left(2^{\frac{1}{\gamma}-1} (M_d)^{\frac{1}{\gamma}} + 2^{\frac{1}{\gamma}-1} (d^{\alpha})^{\frac{1}{\gamma}} \right) \frac{d-Pd}{\left(2^{\frac{1}{\gamma}-1} (M_d)^{\frac{1}{\gamma}} + 2^{\frac{1}{\gamma}-1} (d^{\alpha})^{\frac{1}{\gamma}} \right)} = d \tag{11}$$

for $n \in \mathbb{N}_{n_3}$.

Step 2. T **is continuous.**

By assumption (5) and by definition of B there exists a constant $c > 0$ such that

$$\sum_{i=j}^{\infty} |a_i f(x_{i+1}) + q_i x_i^{\alpha}| \leq c$$

for all $x \in B$. Since $t \to t^{1/\gamma}$ is locally Lipschitz, it is Lipschitz on closed and bounded intervals. There exists a constant L_γ such that

$$\left| t^{1/\gamma} - s^{1/\gamma} \right| \leq L_\gamma |t - s| \text{ for all } t, s \in [-c, c].$$

Since f is locally Lipschitz it is Lipschitz on $[-d, d]$. So there is a constant $L_d > 0$ such that

$$|f(x) - f(y)| \leq L_d |x - y| \tag{12}$$

for all $x, y \in [-d, d]$. Since $x \to x^{\alpha}$ is also Lipschitz on $[-d, d]$, there is a constant L_{α} such that

$$|x^{\alpha} - y^{\alpha}| \leq L_{\alpha} |x - y| \text{ for all } x, y \in [-d, d] .$$

Let $y^{(p)}$ be a sequence in B such that $\left\|x^{(p)} - x\right\| \to 0$ as $p \to \infty$. Since B is closed, $x \in B$. By definition of T, see (10), and by (11) we get

$$\left|(Tx)_n - (Ty^{(p)})_n\right| \leq |p_n| \left|x_{n-k} - y^{(p)}_{n-k}\right|$$

$$+ \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \left| \left[\sum_{i=j}^{\infty} \left(a_i f(x_{i+1}) + q_i \left(x_i\right)^{\alpha}\right)\right]^{\frac{1}{\gamma}} - \left[\sum_{i=j}^{\infty} \left(a_i f(y^{(p)}_{i+1}) + q_i \left(y^{(p)}_i\right)^{\alpha}\right)\right]^{\frac{1}{\gamma}} \right|$$

$$\leq |p_n| \left|x_{n-k} - y^{(p)}_{n-k}\right|$$

$$+ \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} L_{\gamma} \left|\sum_{i=j}^{\infty} a_i f(x_{i+1}) + \sum_{i=j}^{\infty} q_i \left(x_i\right)^{\alpha} - \sum_{i=j}^{\infty} a_i f(y^{(p)}_{i+1}) - \sum_{i=j}^{\infty} q_i \left(y^{(p)}_i\right)^{\alpha}\right|$$

$$\leq |p_n| \left|x_{n-k} - y^{(p)}_{n-k}\right| + L_{\gamma} \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i| \left|f(x_{i+1}) - f(y^{(p)}_{i+1})\right|$$

$$+ L_{\gamma} \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |q_i| \left|(x_i)^{\alpha} - \left(y^{(p)}_i\right)^{\alpha}\right|$$

$$\leq |p_n| \left|x_{n-k} - y^{(p)}_{n-k}\right|$$

$$+ L_{\gamma} L_d \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i| \left|x_{i+1} - y^{(p)}_{i+1}\right| + L_{\gamma} L_{\alpha} \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |q_i| \left|x_i - y^{(p)}_i\right|$$

$$\leq \left(|p_n| + L_{\gamma} L_d \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i| + L_{\gamma} L_{\alpha} \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |q_i|\right) \left\|y^{(p)} - x\right\| .$$

Thus

$$\lim_{p \to \infty} \left\|Ty^{(p)} - Tx\right\| = 0 \text{ as } \lim_{p \to \infty} \left\|y^{(p)} - x\right\| = 0.$$

This means that T is continuous.

Step 3. Comparison of the measure of noncompactness

Now, we need to compare a measure of noncompactness of any subset X of B and $T(X)$. Let us fix any nonempty set $X \subset B$. Take any sequences $x, y \in X$. Following the same calculations which led to the continuity of the operator T we see that

$$|(Tx)_n - (Ty)_n| \leq (|p_n| + L_\gamma L_\alpha \beta_n) |x_n - y_n| + L_\gamma L_d \alpha_n |x_{n+1} - y_{n+1}|$$

Taking sufficiently large n we get $|p_n| + L_\gamma L_d \alpha_n \leq c_1 < \frac{1}{2}$ and $L_\gamma L_\alpha \beta_n \leq c_2 < \frac{1}{2}$ since $\alpha_n \to 0$ and $\beta_n \to 0$, where c_1, c_2 are some constants. We see that

$$diam\ (T(X))_n \leq c_1 diam\ X_n + c_2 diam\ X_{n+1}.$$

This yields by the properties of the upper limit that

$$\limsup_{n\to\infty} diam\ (T(X))_n \leq c_1 \limsup_{n\to\infty} diam\ X_n + c_2 \limsup_{n\to\infty} diam\ X_{n+1}.$$

From above, for any $X \subset B$ we have $\mu(T(X)) \leq (c_1 + c_2)\, \mu(X)$.

Step 2. Relation between fixed points and solutions

By Theorem 1 we conclude that T has a fixed point in the set B. It means that there exists $x \in B$ such that

$$x_n = (Tx)_n.$$

Thus

$$x_n = -p_n x_{n-k} - \sum_{j=n}^{\infty} \left[\frac{1}{r_j} \sum_{i=j}^{\infty} (a_i f(x_{i+1}) + q_i x_i^\alpha) \right]^{\frac{1}{\gamma}}, \text{ for } n \in \mathbb{N}_{n_3} \tag{13}$$

To show that there exists a correspondence between fixed points of T and solutions to (1) we apply operator Δ to both sides of the following equation

$$x_n + p_n x_{n-k} = -\sum_{j=n}^{\infty} \left[\frac{1}{r_j} \sum_{i=j}^{\infty} (a_i f(x_{i+1}) + q_i x_i^\alpha) \right]^{\frac{1}{\gamma}},$$

which is obtained from (13). We find that

$$\Delta(x_n + p_n x_{n-k}) = -\left[\frac{1}{r_n} \sum_{i=n}^{\infty} (a_i f(x_{i+1}) + q_i x_i^\alpha) \right]^{\frac{1}{\gamma}}, \ n \in \mathbb{N}_{n_3}.$$

and next

$$(\Delta(x_n + p_n x_{n-k}))^\gamma = -\frac{1}{r_n} \sum_{i=n}^{\infty} \left(a_i f(x_i) + q_i x_i^\alpha\right), \ \ n \in \mathbb{N}_{n_3}.$$

Taking operator Δ again to both sides of the above equation we obtain

$$\Delta \left(r_n \left(\Delta(x_n + p_n x_{n-k})\right)^\gamma\right) = -a_n f(x_{n+1}) - q_n x_n^\alpha, \ \ n \in \mathbb{N}_{n_3}.$$

Using again the operator Δ for both sides of the above equation and taking the γ power, we get equation (1) for $n \in \mathbb{N}_{n_3}$. Sequence x, which is a fixed point of mapping T, is a bounded sequence which fulfills equation (1) for large n. If $n_3 \leq k$ the proof is ended. If $n_3 > k$ we find previous $n_3 - k + 1$ terms of sequence x by formula

$$x_{n-k+l} = \frac{1}{p_{n+l}}(-x_{n+l} + \sum_{j=n+l}^{\infty} \left[\frac{1}{r_j} \sum_{i=j}^{\infty} (a_i f(x_{i+1}) + q_i x_i^\alpha)\right]^{\frac{1}{\gamma}}),$$

where $l \in \{0, 1, 2, \ldots, k-1\}$, which results leads directly from (1). It means that equation (1) has at least one bounded solution $x : \mathbb{N}_k \to \mathbb{R}$.

This completes the proof. ■

4 Lyapunov type stability

In this section we derive sufficient conditions for the existence of an asymptotically stable solution of equation (1). We recall the following definition which can be found in [6].

Definition 2. *Let x is real a function defined, bounded and continuous on $[0, \infty)$. The function x is an asymptotically stable solution of equation*

$$x = Fx \tag{14}$$

it means that for any $\varepsilon > 0$ there exists $T > 0$ such that for every $t \geq T$ and for every other solution y of equation (14) the following inequality holds

$$|x(t) - y(t)| \leq \varepsilon.$$

Theorem 3. *Assume that the ratio of odd positive integers $\gamma \in (0, 1]$, number $\alpha \in (0, 1)$ is fixed and that k is a fixed positive integer. Assume further that there exists a positive constant D such that*

$$|f(x) - f(y)| \leq D\,|x - y| \tag{15}$$

for any $x, y \in \mathbb{R}$, and conditions (3)-(5) *hold. Then equation* (1) *has at least one asymptotically stable solution $x : \mathbb{N}_k \to \mathbb{R}$.*

Proof. From Theorem 2, equation (1) has at least one bounded solution $x : \mathbb{N}_0 \to \mathbb{R}$ which can be rewritten in the form

$$x_n = (Tx)_n, \tag{16}$$

where mapping T is defined by (10). By Definition 2, sequence x is an asymptotically stable solution of equation $x_n = (Tx)_n$ means that for any $\varepsilon > 0$ there exists $n_4 \in \mathbb{N}_0$ such that for every $n \geq n_4$ and for every other solution y of equation (1) the following inequality holds

$$|x_n - y_n| \leq \varepsilon. \tag{17}$$

Note that since $\alpha \in (0, 1)$ it follows that function $t \to t^\alpha$ is Lipschitz on $\mathbb{R}$. Then, following the steps of the proof of Theorem 2 we see that

$$|x_n - y_n| = |(Tx)_n - (Ty)_n| \leq$$

$$|p_n| \, |x_{n-k} - y_{n-k}| + L_\gamma D \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i| \, |x_{i+1} - y_{i+1}| +$$

$$L_\gamma L_\alpha \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} q_i \, |x_i - y_i|$$

Note that for n large enough, say $n \geq n_4 \geq n_3$ we have

$$\vartheta := |p_n| + L_\gamma D \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} |a_i| + L_\gamma L_\alpha \sum_{j=n}^{\infty} \left|\frac{1}{r_j}\right|^{\frac{1}{\gamma}} \sum_{i=j}^{\infty} q_i < 1$$

Let us denote

$$\limsup_{n\to\infty} |x_n - y_n| = l,$$

and observe that

$$\limsup_{n\to\infty} |x_n - y_n| = \limsup_{n\to\infty} |x_{n-k} - y_{n-k}| = \limsup_{n\to\infty} |x_{n+1} - y_{n+1}| .$$

Thus from the above we have

$$l \leq \vartheta \cdot l.$$

This means that $\limsup_{n\to\infty} |x_n - y_n| = 0$. This completes the proof. ■

5 Comments and an example

In [15] the authors consider a special type of problem (1), namely they investigate the existence of a solution and Lyapunov type stability to the following equation

$$\Delta\left(r_n \Delta x_n\right) = a_n f(x_{n+1}). \tag{18}$$

Their main assumption is the linear growth assumption on nonlinear term f. More precisely, they assume that there exists a positive constant M such that $|f(x_n)| \leq M\ |x_n|$ for all $x \in N_0$. Using ideas developed in this paper we get the following result.

Theorem 4. *Assume that*

$$f : \mathbb{R} \to \mathbb{R} \textit{ is a locally Lipschitz function,}$$

and the sequences $r : \mathbb{N}_0 \to \mathbb{R} \setminus \{0\}$, $a : \mathbb{N}_0 \to \mathbb{R}$ *are such that*

$$\sum_{n=0}^{\infty} \left|\frac{1}{r_j}\right| \sum_{i=n}^{\infty} |a_i| < +\infty.$$

Let the sequence $p : \mathbb{N}_0 \to \mathbb{R}\setminus\{0\}$ *satisfy the condition* (4)*.Then, there exists a bounded solution* $x : \mathbb{N}_0 \to \mathbb{R}$ *of equation* (18)*.*

Finally, we give an example of equation which can be considered by our method.

Example 1. Take $k = 3$, an arbitrary C^1 function $f : \mathbb{R} \to \mathbb{R}$ and consider the following problem

$$\Delta\left(\left(-1\right)^n \Delta\left(x_n + \frac{1}{2}x_{n-3}\right)^{1/3}\right) + \frac{1}{2^n}\left(x_n\right)^5 + f\left(x_{n+1}\right) = 0. \tag{19}$$

Taking $\gamma = \frac{1}{3}$, $\alpha = 5$, $r_n = (-1)^n$, $p_n = \frac{1}{2}$, $a_n = q_n = \frac{1}{2^n}$ with $f(x) = \sqrt[5]{x}$ we see that $x_n = (-1)^n$ is a bounded solution to (19). By Theorem 3 this solution is asymptotically stable.

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